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universal; but the expression is the result of very different operations in the two systems. Modern mathematics has given new meanings to old symbols, or, more properly speaking, it has used old symbols to represent new ideas, and thus the same symbol has a double meaning—and in some cases several meanings. Thus, in algebra, dx implies that x is multiplied by d , but in calculus d implies an operation; in the expression d^2x , the figure 2 may imply a square or a second operation; -1 in the expression a^{-1} implies a reciprocal, but in the expression $\sin^{-1}x$ an inverse function; the period (.) is used to denote a decimal fraction, multiplication, or a fluxion; $\beta = 0$ in algebra means that β is zero, but in modern geometry it may represent the equation of a line; &c. Conversely, the same symbol may be put under different forms; thus we have

$$a \div b = \frac{a}{b}; a^{-1} = \frac{1}{a}; dx = x; \sqrt{-1} = (-1)^{\frac{1}{2}}; \&c.$$

Hamilton's system of quaternions and the imaginaries of algebra have a relation to each other similar to that between Chemistry and Alchemy. The false science of alchemy gave birth to the true science of chemistry,* but the latter, instead of solving the problem of the former—the transmutation of metals into gold—shows it to be all the more impossible, and has entered new fields of inquiry; so the imaginaries of algebra gave rise to the real system of Quaternions, but instead of geometrizing the former leaves them as imaginary as before and enters its own field of inquiry by developing new processes of analysis.

As soon as chemistry was established there was a deluge of pretenders in science who *attempted* to find the philosopher's stone; so after the establishment of the science of quaternions, there were, and still are, some who hold that *all* imaginaries are so only in name, but are in fact real quantities.

(To be continued.)

DETERMINATION OF A SPHERE WHICH CUTS FIVE GIVEN SPHERES AT THE SAME ANGLE.

BY DR. THOMAS CRAIG, WASHINGTON, D. C.

GENERAL Benjamin Alvord a few days ago attracted my attention to this problem as being one which had, at one time, possessed a peculiar interest

*J. Andrew, in the ninth edition of the Encyclopædia Britannica, says, "Alchemy was, we may say, the sickly but imaginative infancy through which modern chemistry had to pass before it attained its majority."

for him on account of its connection with the large class of problems on intersections and tangencies to which he has given so much attention, and from which he has obtained so many valuable results by the elegant processes of synthetic geometry. Gen. Alvord at the same time informed me that he had communicated the problem to the ANALYST, through the agency of Marcus Baker, Esq., and that Mr. Adcock, in the Sept. number of the ANALYST for 1877, had furnished the equations necessary for its solution, but did not give the solution.

Having given the necessary equations, I presume that Mr. Adcock thought it mere journeyman's work to obtain from them the required forms of the unknown quantities. The following solution is extremely simple but I have thought that it might not be destitute of interest to the readers of the ANALYST on account of the symmetry of the expressions and the ease with which the eliminations can be performed.

Call θ the angle of intersection and $(x_1, y_1, z_1), (x_2, y_2, z_2)$ &c., the coordinates of the centers of the given spheres, r_1, r_2, r_3, r_4, r_5 their radii, ρ the radius of the required sphere and (α, β, γ) the coordinates of its center, then we have by equating the expressions for the distances between the centers of any one sphere and the required sphere, in polar and rectangular coordinates

$$\cos \theta = \frac{(x_i - \alpha)^2 + (y_i - \beta)^2 + (z_i - \gamma)^2 - r_i^2 - \rho^2}{2r_i \rho}, \quad (1)$$

where i has the values 1, 2, 3, 4, 5.

These are the expressions given by Mr. Adcock for the determination of the four unknown quantities $(\alpha, \beta, \gamma, \rho)$; after these are determined simple substitution will give a value of $\cos \theta$ for any given system of five spheres.

For the determination of $(\alpha, \beta, \gamma, \rho)$ we have the equations

$$\begin{aligned} r_2[(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - r_1^2 - \rho^2] &= r_1[(x_2 - \alpha)^2 + (y_2 - \beta)^2 + (z_2 - \gamma)^2 - r_2^2 - \rho^2], \\ r_3[(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - r_1^2 - \rho^2] &= r_1[(x_3 - \alpha)^2 + (y_3 - \beta)^2 + (z_3 - \gamma)^2 - r_3^2 - \rho^2], \\ r_4[(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - r_1^2 - \rho^2] &= r_1[(x_4 - \alpha)^2 + (y_4 - \beta)^2 + (z_4 - \gamma)^2 - r_4^2 - \rho^2], \\ r_5[(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - r_1^2 - \rho^2] &= r_1[(x_5 - \alpha)^2 + (y_5 - \beta)^2 + (z_5 - \gamma)^2 - r_5^2 - \rho^2]. \end{aligned} \quad (2)$$

Expand these expressions and for brevity write

$$\begin{aligned} r_1 r_2 (r_2 - r_1) + r_2 (x_1^2 + y_1^2 + z_1^2) - r_1 (x_2^2 + y_2^2 + z_2^2) &= P_1, \\ \vdots & \quad \quad \quad \vdots \\ r_1 r_5 (r_5 - r_1) + r_5 (x_1^2 + y_1^2 + z_1^2) - r_1 (x_5^2 + y_5^2 + z_5^2) &= P_5, \end{aligned}$$

P_1, P_2 &c., being of course determinate constants. We find now

$$\begin{aligned}
 (r_2-r_1)(\alpha^2+\beta^2+\gamma^2-\rho^2)-2[(r_2x_1-r_1x_2)\alpha+(r_2y_1-r_1y_2)\beta+(r_2z_1-r_1z_2)\gamma] &= P_1 \\
 (r_3-r_1)(\alpha^2+\beta^2+\gamma^2-\rho^2)-2[(r_3x_1-r_1x_3)\alpha+(r_3y_1-r_1y_3)\beta+(r_3z_1-r_1z_3)\gamma] &= P_2 \\
 (r_4-r_1)(\alpha^2+\beta^2+\gamma^2-\rho^2)-2[(r_4x_1-r_1x_4)\alpha+(r_4y_1-r_1y_4)\beta+(r_4z_1-r_1z_4)\gamma] &= P_3 \\
 (r_5-r_1)(\alpha^2+\beta^2+\gamma^2-\rho^2)-2[(r_5x_1-r_1x_5)\alpha+(r_5y_1-r_1y_5)\beta+(r_5z_1-r_1z_5)\gamma] &= P_4
 \end{aligned} \tag{3}$$

Divide these through by r_2-r_1 , r_3-r_1 , r_4-r_1 , r_5-r_1 respectively and place

$$\frac{P_1}{r_2-r_1} = Q_1, \quad \frac{P_2}{r_3-r_1} = Q_2 \text{ \&c.};$$

then subtract the second equation from the first, the third from the second and so on: by this means we eliminate the quantity

$$\alpha^2 + \beta^2 + \gamma^2 - \rho^2$$

and in fact eliminate ρ altogether and have left three equations containing three unknown quantities α, β, γ .

The result of the first substitution is of the form

$$\frac{[r_1^2(x_3-x_2)+r_1r_2(x_1-x_3)+r_1r_3(x_2-x_1)]\alpha}{(r_3-r_1)(r_2-r_1)} + \dots = Q_1 - Q_2,$$

the terms in β and γ are obtained by merely changing x into y and z respectively. The next equations have their first terms in the forms

$$\frac{[r_1^2(x_4-x_3)+r_1r_3(x_1-x_4)+r_1r_4(x_3-x_1)]\alpha}{(r_3-r_1)(r_4-r_1)} + \dots = Q_2 - Q_3,$$

$$\frac{[r_1^2(x_5-x_4)+r_1r_4(x_1-x_5)+r_1r_5(x_4-x_1)]\alpha}{(r_4-r_1)(r_5-r_1)} + \dots = Q_3 - Q_4.$$

These equations can all be given in the determinant form as follows:

$$\begin{aligned}
 \begin{vmatrix} r_1 & r_2 & r_3 \\ x_1 & x_2 & x_3 \\ r_1 & r_1 & r_1 \end{vmatrix} \alpha + \begin{vmatrix} r_1 & r_2 & r_3 \\ y_1 & y_2 & y_3 \\ r_1 & r_1 & r_1 \end{vmatrix} \beta + \begin{vmatrix} r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ r_1 & r_1 & r_1 \end{vmatrix} \gamma &= (Q_2 - Q_1)(r_2 - r_1)(r_3 - r_1), \\
 \begin{vmatrix} r_1 & r_3 & r_4 \\ x_1 & x_3 & x_4 \\ r_1 & r_1 & r_1 \end{vmatrix} \alpha + \begin{vmatrix} r_1 & r_3 & r_4 \\ y_1 & y_3 & y_4 \\ r_1 & r_1 & r_1 \end{vmatrix} \beta + \begin{vmatrix} r_1 & r_3 & r_4 \\ z_1 & z_3 & z_4 \\ r_1 & r_1 & r_1 \end{vmatrix} \gamma &= (Q_3 - Q_2)(r_3 - r_1)(r_4 - r_1), \\
 \begin{vmatrix} r_1 & r_4 & r_5 \\ x_1 & x_4 & x_5 \\ r_1 & r_1 & r_1 \end{vmatrix} \alpha + \begin{vmatrix} r_1 & r_4 & r_5 \\ y_1 & y_4 & y_5 \\ r_1 & r_1 & r_1 \end{vmatrix} \beta + \begin{vmatrix} r_1 & r_4 & r_5 \\ z_1 & z_4 & z_5 \\ r_1 & r_1 & r_1 \end{vmatrix} \gamma &= (Q_4 - Q_3)(r_4 - r_1)(r_5 - r_1).
 \end{aligned} \tag{4}$$

The coefficients of α, β, γ are constants and of course can be readily computed, for any particular case, by merely expanding these determinants; the same is true of the quantities on the right hand side of the equations, they are constants and readily computed. Call the values of the determinants in the first second and third equations, respectively,

$$\begin{array}{ccc} A_1, & B_1, & C_1, \\ A_2, & B_2, & C_2, \\ A_3, & B_3, & C_3, \end{array}$$

and denote the second members of the equations by

$$D_1, \quad D_2, \quad D_3,$$

respectively, then these are

$$\begin{array}{l} A_1\alpha + B_1\beta + C_1\gamma = D_1, \\ A_2\alpha + B_2\beta + C_2\gamma = D_2, \\ A_3\alpha + B_3\beta + C_3\gamma = D_3; \end{array} \quad (5)$$

writing for brevity Δ = the determinants of the coefficients A, B, C , i. e.,

$$\Delta = \begin{vmatrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ A_3, & B_3, & C_3 \end{vmatrix}$$

we have by direct solution

$$\left. \begin{array}{l} \alpha = \frac{1}{\Delta} \begin{vmatrix} D_1, & B_1, & C_1 \\ D_2, & B_2, & C_2 \\ D_3, & B_3, & C_3 \end{vmatrix}, \\ \beta = \frac{1}{\Delta} \begin{vmatrix} C_1, & D_1, & A_1 \\ C_2, & D_2, & A_2 \\ C_3, & D_3, & A_3 \end{vmatrix}, \\ \gamma = \frac{1}{\Delta} \begin{vmatrix} A_1, & B_1, & D_1 \\ A_2, & B_2, & D_2 \\ A_3, & B_3, & D_3 \end{vmatrix}. \end{array} \right\} \quad (6)$$

Nothing in the second members of these equations but known quantities, therefore the center of the sphere is determined. In order to find the radius ρ , it is only necessary to substitute these values of α, β, γ in any one of equations (2), and solve the result for ρ^2 . In fact we have

$$\rho^2 = \frac{r_2[(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2] - r_1[(x_2 - \alpha)^2 + (y_2 - \beta)^2 + (z_2 - \gamma)^2]}{r_2 - r_1}.$$

The position of the center and the radius of the required sphere are now found. If it is desired to find at what angle the sphere intersects the five given ones, the quantities α, β, γ and ρ must be substituted in equation (1) and the result will be the cosine of the required angle.

It is worth while to observe that the coefficients of α, β, γ in the preceding linear equations are all derived from the same determinant, are in fact second minors of

$$V \equiv \begin{vmatrix} r_1, & r_2, & r_3, & r_4, & r_5 \\ x_1, & x_2, & x_3, & x_4, & x_5 \\ y_1, & y_2, & y_3, & y_4, & y_5 \\ z_1, & z_2, & z_3, & z_4, & z_5 \\ r_1, & r_1, & r_1, & r_1, & r_1 \end{vmatrix}$$

This consideration would enable us to replace A, B, C by certain second differential coefficients of V , but it is not necessary to go into that.